Formalizing the Isoperimetric Inequality in Lean 4

Miraj Samarakkody

Department of Mathematics and Computer Science, Tougaloo College, Tougaloo, MS 39174.

smsamarakkody@tougaloo.edu

Historical Background

The isoperimetric problem, determining which plane figure with a given perimeter encloses the maximum area, has captivated mathematicians since antiquity. The answer, the circle, is referenced as far back as Virgil's *Aeneid* through the legend of Queen Dido (ca. 800 BCE). Around 200 BCE, Zenodorus proved that a circle encloses more area than any regular polygon of equal perimeter. Rigorous proofs emerged only in the 19th century: Jakob Steiner (1838) gave ingenious but incomplete arguments, and Karl Weierstrass (ca. 1870) provided the first rigorous proof via the calculus of variations. In 1902, Adolf Hurwitz discovered an elegant proof using Fourier analysis. This work presents a formal verification of the classical isoperimetric inequality $L^2 \geq 4\pi A$ in Lean 4 and its mathematical library Mathlib, following Hurwitz's Fourier-analytic approach.

Motivation: Formal Proofs in the Age of AI

Large language models have transformed how people interact with mathematics, yet even state-of-the-art AI systems routinely produce *plausible-looking but incorrect* mathematical arguments. They confuse quantifiers, skip case analyses, and generate proofs that pass casual inspection but fail under scrutiny. The core difficulty is that fluency in mathematical language is not the same as *soundness*. Formal proof assistants such as Lean 4 provide a solution: every logical step is machine-checked against a foundational type theory, making silent errors impossible. Beyond correctness guarantees, formally verified proofs are **high-quality training data** for the next generation of AI reasoning systems. Unlike natural-language mathematics, formal proofs carry explicit logical structure, dependency graphs, and machine-checkable certificates, precisely the signal needed to teach AI models what a *valid* mathematical argument actually looks like. This work contributes a fully verified proof of the classical isoperimetric inequality $L^2 \geq 4\pi A$ to this growing corpus, bridging classical analysis and modern proof engineering in service of both human and machine mathematical reasoning.

Main Result

Theorem (Isoperimetric Inequality)

Let C be a simple closed curve in the plane with perimeter L and enclosed area A. Then

$$L^2 \geq 4\pi A,$$

with equality if and only if C is a circle.

The formalization proceeds in two phases:

1. **Fourier-analytic foundations:** trigonometric orthogonality, Parseval's theorem, the Weierstrass M-test, and Wirtinger's inequality.
2. **Hurwitz's proof:** reparametrization, the shoelace formula, integration by parts, and the AM–GM inequality.

The complete Lean 4 formalization is available at
`https://github.com/mirajcs/IsoperimetricInequality`

Fourier-Analytic Foundations

For a 2π-periodic function f, the classical Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

Lemma (Trigonometric Orthogonality)

For $n, m \in \mathbb{N}^+$:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \pi\,\delta_{nm}, \quad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \pi\,\delta_{nm},$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

Theorem (Parseval's Theorem)

Under appropriate integrability and summability conditions,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \big[f(x)\big]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\big(a_n^2 + b_n^2\big).$$

Theorem (Wirtinger's Inequality)

Let f be a 2π-periodic function with continuous derivative and $\int_0^{2\pi} f(x)\,dx = 0$. Then

$$\int_0^{2\pi} \big[f(x)\big]^2\,dx \;\leq\; \int_0^{2\pi} \big[f'(x)\big]^2\,dx,$$

with equality if and only if $f(x) = a\cos x + b\sin x$.

Proof sketch. The zero-mean condition gives $a_0 = 0$, so Parseval applied to f and f' yields

$$\frac{1}{\pi}\int [f]^2 = \sum (a_n^2 + b_n^2), \qquad \frac{1}{\pi}\int [f']^2 = \sum n^2 (a_n^2 + b_n^2).$$

Since $n \geq 1$ implies $n^2 \geq 1$, the inequality follows.

Hurwitz's Proof

Given a simple closed C^1 curve γ of length L, reparametrize: set $f(\theta) = x\left(\frac{L\theta}{2\pi}\right)$, $g(\theta) = y\left(\frac{L\theta}{2\pi}\right)$, so that f, g are 2π-periodic. The proof proceeds in five steps.

1. **Shoelace formula.** The enclosed area satisfies
$$A = \frac{1}{2}\int_0^{2\pi} \big[f(\theta)g'(\theta) - g(\theta)f'(\theta)\big]\,d\theta = \int_0^{2\pi} f(\theta)\,g'(\theta)\,d\theta.$$
(Integration by parts uses periodicity to cancel boundary terms.)
2. **AM–GM inequality.** For any u, v: $uv \leq \frac{1}{2}(u^2 + v^2)$. Hence
$$2A \;=\; 2\int_0^{2\pi} f\,g'\,d\theta \;\leq\; \int_0^{2\pi} \big(f^2 + (g')^2\big)\,d\theta.$$
3. **Wirtinger's inequality.** Assuming $\int_0^{2\pi} f\,d\theta = 0$:
$$\int_0^{2\pi} f^2\,d\theta \;\leq\; \int_0^{2\pi} (f')^2\,d\theta.$$
This gives $2A \leq \int_0^{2\pi} [(f')^2 + (g')^2]\,d\theta$.
4. **Arc-length constraint.** Unit-speed parametrization gives $(f')^2 + (g')^2 = \left(\frac{L}{2\pi}\right)^2$, so
$$\int_0^{2\pi} \big[(f')^2 + (g')^2\big]\,d\theta = \frac{L^2}{2\pi}.$$
5. **Conclusion.**
$$2A \;\leq\; \frac{L^2}{2\pi}, \quad\Longrightarrow\quad L^2 \geq 4\pi A. \qquad \square$$

Equality holds iff f and g' are proportional, which (via Wirtinger's equality condition) characterizes circles.

Formalization in Lean 4

The formalization uses Lean 4 and Mathlib, based on dependent type theory (the Calculus of Inductive Constructions), where propositions are types and proofs are terms.

Simple Closed Curves

A simple closed C^1 curve in $\mathbb{R}^n$ is encoded as a structure with fields:

- `curve`: $\mathbb{R} \to$ `EuclideanSpace` $\mathbb{R}$ (`Fin` n)
- Positive arc length, continuity, C^1 differentiability
- Closure: $\gamma(0) = \gamma(L)$ and periodicity: $\gamma(t + L) = \gamma(t)$
- Injectivity on $(0, L)$ (simplicity)

Key Formalization Challenges

- **Sum–integral interchange:** The Dominated Convergence Theorem (`MeasureTheory.integral_tsum_of_summable_norm`) requires explicit L^1 summability bounds.
- **Term-by-term differentiation:** Requires convergence of both $\sum(\|a_n\| + \|b_n\|)$ and $\sum n(\|a_n\| + \|b_n\|)$; handled by `hasDerivAt_of_tendstoUniformly`.

Open Problems

1. **Closing the Parseval hypothesis:** Derive the orthogonality-based identity $\int [f(x)]^2\,dx = \pi\sum(a_n^2 + b_n^2)$ from first principles within the formalization.
2. **Equality case:** Formally verify that $L^2 = 4\pi A$ if and only if C is a circle.
3. **Lower regularity:** Extend the result to C^0 or Lipschitz curves.
4. **Higher dimensions:** Formalize the isoperimetric inequality in $\mathbb{R}^n$.

Acknowledgement

This work is supported by the HBCU-UP Implementation project, NSF Award No. 2510537.

References

[1] A. Hurwitz, **Sur quelques applications géométriques des séries de Fourier**, *Ann. Sci. Éc. Norm. Supér.* **19** (1902), 357–408.

[2] K. Weierstrass, **Mathematische Werke**, *Mayer & Müller*, Berlin, 1895.

[3] T. Bonnesen, W. Fenchel, **Theory of Convex Bodies**, *BCS Associates*, 1987.

[4] L. de Moura, S. Ullrich, **The Lean 4 Theorem Prover and Programming Language**, *Proc. CADE-28*, LNAI **12699** (2021), 625–635.

[5] The Mathlib Community, **The Lean Mathematical Library**, *Proc. CPP 2020*, ACM, 367–381.

[6] M. Samarakkody, **Formalizing the Classical Isoperimetric Inequality in the Two-Dimensional Case**, *Submitted* (2026). arXiv:2603.14663