Lecture Notes: The Chain Rule

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Table of contents

1. Where we are . . . 1
2. The Chain Rule . . . 2
3. The Power Rule combined with the Chain Rule . . . 4
4. Exponential functions . . . 5
5. Longer chains . . . 6
6. A numerical sanity check . . . 7
7. Why the Chain Rule is true . . . 8
8. Practice problems . . . 8
Summary . . . 9

Stewart, ***Calculus****, 9th ed., §3.4*

Learning objectives. By the end of this lecture you should be able to: 1. Recognize a **composite function** $F = f \circ g$ and identify its outer function f and inner function g. 2. State the **Chain Rule** in both prime and Leibniz notation, and interpret it as multiplying rates of change. 3. Use the **Power Rule combined with the Chain Rule** to differentiate $[g(x)]^n$. 4. Differentiate b^x for any base $b > 0$. 5. Apply the Chain Rule several times to longer chains, and together with the Product and Quotient Rules. 6. Check every answer symbolically with **SymPy** and numerically with **NumPy**.

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import Math, display

sp.init_printing()
x, t, u, theta = sp.symbols("x t u theta")
```

1. Where we are

From §3.1–3.3 we can differentiate:

Function	Derivative
x^n	nx^{n-1}
e^x	e^x
$\sin x,\ \cos x,\ \tan x$	$\cos x,\ -\sin x,\ \sec^2 x$
$f\,g$	$fg' + gf'$
$\dfrac{f}{g}$	$\dfrac{gf' - fg'}{g^2}$

None of these rules can handle

$$F(x) = \sqrt{x^2 + 1}.$$

F is a **composite function**: if $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$, then $y = F(x) = f(g(x))$, so $F = f \circ g$. We call f the **outer** function and g the **inner** function. We already know how to differentiate f and g. The Chain Rule tells us how to put those derivatives together.

As before, the helper below compares a derivative found by hand with SymPy's answer.

```
def check(f, by_hand, var=x):
    """Display d/dvar f = by_hand and confirm it agrees with SymPy."""
    ok = sp.simplify(sp.diff(f, var) - by_hand) == 0
    display(Math(rf"\frac{{d}}{{d{sp.latex(var)}}}\left({sp.latex(f)}\right) = {sp.latex(by_hand)}"))
    print("Agrees with SymPy:", ok)
```

2. The Chain Rule

Motivation: rates multiply. Suppose u changes twice as fast as x, and y changes three times as fast as u. Then y changes $3 \cdot 2 = 6$ times as fast as x:

$$\frac{du}{dx} = 2, \quad \frac{dy}{du} = 3 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6.$$

Think of three gears: if gear A turns x times, gear B turns u times, and gear C turns y times, the gear ratios multiply.

The Chain Rule. If g is differentiable at x and f is differentiable at $g(x)$, then $F = f \circ g$ is differentiable at x and

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

In words: *the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.*

Warning: in $f'(g(x))$ the inner function is left **unchanged** inside f'. Differentiate the outside first, keep the inside as it is, then multiply by the derivative of the inside.

Leibniz notation is a good memory aid, but du does not really "cancel." $\dfrac{dy}{du}$ is not a fraction. The rule still has to be proved (see §7).

We check the rule for general functions f and g:

```
f, g = sp.Function("f"), sp.Function("g")
lhs = sp.diff(f(g(x)), x)
rhs = f(u).diff(u).subs(u, g(x)) * sp.diff(g(x), x)
display(Math(rf"\frac{{d}}{{dx}}f(g(x)) = {sp.latex(lhs)}"))
print("Chain rule holds:", sp.simplify(lhs - rhs) == 0)
```

$$\frac{d}{dx}f(g(x)) = \frac{d}{dg(x)}f(g(x))\frac{d}{dx}g(x)$$

```
Chain rule holds: True
```

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

By hand. The outer function is $f(u) = \sqrt{u} = u^{1/2}$, so $f'(u) = \dfrac{1}{2\sqrt{u}}$. The inner function is $g(x) = x^2 + 1$, so $g'(x) = 2x$.

$$F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

In Leibniz notation. Let $u = x^2 + 1$ and $y = \sqrt{u}$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

```
check(sp.sqrt(x**2 + 1), x / sp.sqrt(x**2 + 1))
```

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{x}{\sqrt{x^2+1}}$$

```
Agrees with SymPy: True
```

Example 2. Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

These look alike but are different compositions.

(a) The outer function is sin, the inner function is x^2:

$$\frac{dy}{dx} = \underbrace{\cos(x^2)}_{\text{outer}' \text{ at inner}} \cdot \underbrace{2x}_{\text{inner}'} = 2x\cos(x^2).$$

(b) $\sin^2 x$ means $(\sin x)^2$. The outer function is squaring, the inner function is sin:

$$\frac{dy}{dx} = 2(\sin x)\cdot\cos x = 2\sin x\cos x = \sin 2x.$$

```
check(sp.sin(x**2), 2*x * sp.cos(x**2))
check(sp.sin(x)**2, sp.sin(2*x))
```

$$\frac{d}{dx}\left(\sin\left(x^2\right)\right) = 2x\cos\left(x^2\right)$$

```
Agrees with SymPy: True
```

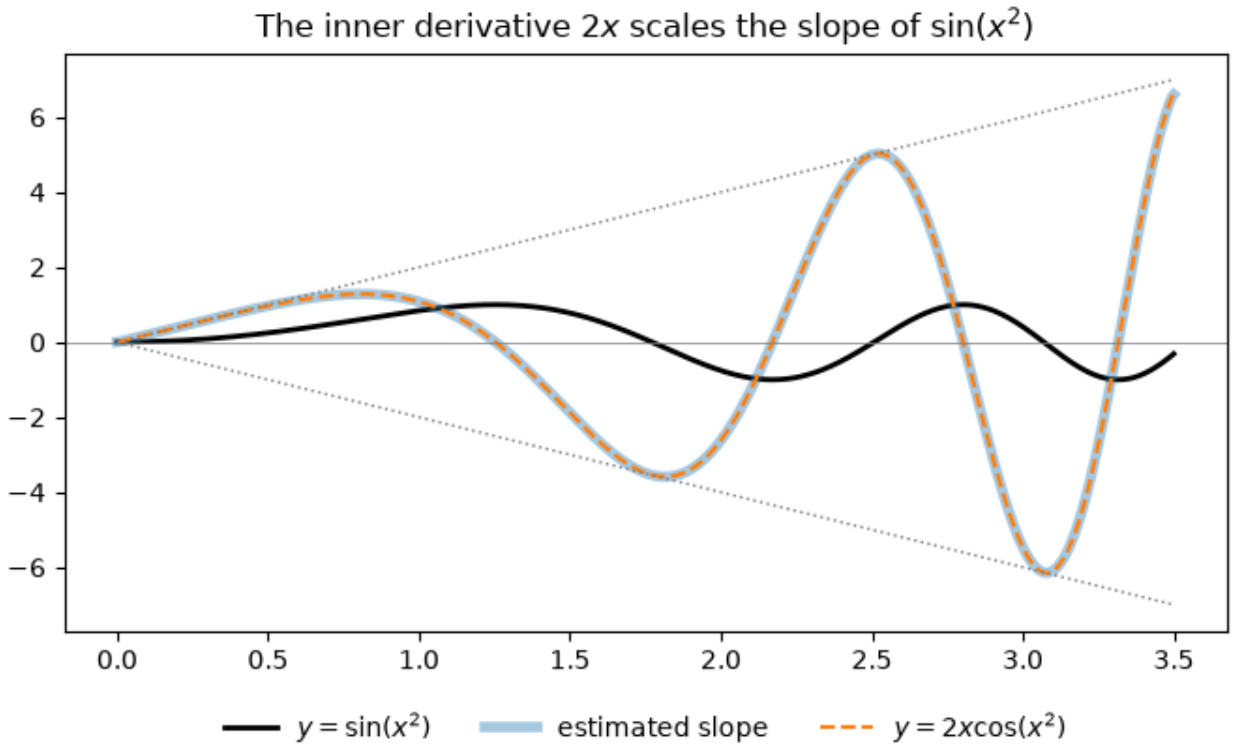
$$\frac{d}{dx}\left(\sin^2\left(x\right)\right) = \sin\left(2x\right)$$

```
Agrees with SymPy: True
```

The plot shows why the factor $2x$ appears in (a). As $|x|$ grows, x^2 grows faster and faster, so $\sin(x^2)$ oscillates faster and faster. Its slopes get steeper, and the derivative $2x\cos(x^2)$ has a growing envelope $\pm 2x$.

```
xs = np.linspace(0, 3.5, 600)
slopes = np.gradient(np.sin(xs**2), xs)  # numerical estimate of the slope

fig, ax = plt.subplots(figsize=(8, 4))
ax.plot(xs, np.sin(xs**2), "k", lw=2, label=r"$y = \sin(x^2)$")
ax.plot(xs, slopes, lw=4, alpha=0.4, label=r"estimated slope")
ax.plot(xs, 2 * xs * np.cos(xs**2), "--", lw=1.5, label=r"$y = 2x\cos(x^2)$")
ax.plot(xs, 2 * xs, ":", color="gray", lw=1)
ax.plot(xs, -2 * xs, ":", color="gray", lw=1)
ax.axhline(0, color="gray", lw=0.5)
ax.legend(loc="upper center", bbox_to_anchor=(0.5, -0.1), ncol=3, frameon=False)
ax.set_title(r"The inner derivative $2x$ scales the slope of $\sin(x^2)$")
plt.show()
```



3. The Power Rule combined with the Chain Rule

When the outer function is a power, $y = u^n$ with $u = g(x)$, the Chain Rule gives:

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\, g'(x)} \qquad \text{or} \qquad \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

The ordinary Power Rule is the case $g(x) = x$, where $g'(x) = 1$.

Example 3. Differentiate $y = (x^3 - 1)^{100}$.

By hand. Do **not** expand this. With $u = x^3 - 1$:

$$\frac{dy}{dx} = 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}.$$

```
check((x**3 - 1)**100, 300 * x**2 * (x**3 - 1)**99)
```

$$\frac{d}{dx}\left((x^3 - 1)^{100}\right) = 300x^2\,(x^3 - 1)^{99}$$

```
Agrees with SymPy: True
```

Example 4. Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

By hand. Rewrite as a power first: $f(x) = (x^2 + x + 1)^{-1/3}$. Then

$$f'(x) = -\frac{1}{3}(x^2 + x + 1)^{-4/3}\,(2x + 1) = -\frac{2x + 1}{3(x^2 + x + 1)^{4/3}}.$$

Rewriting as a power avoids the Quotient Rule.

```
check(1 / sp.cbrt(x**2 + x + 1), -(2*x + 1) / (3 * (x**2 + x + 1)**sp.Rational(4, 3)))
```

$$\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2+x+1}}\right) = \frac{-2x-1}{3\left(x^2+x+1\right)^{\frac{4}{3}}}$$

```
Agrees with SymPy: True
```

Example 5. Find the derivative of $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$.

By hand. Power Rule + Chain Rule on the outside, Quotient Rule on the inside:

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{(2t+1)\cdot 1 - (t-2)\cdot 2}{(2t+1)^2} = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}.$$

```
check(((t - 2) / (2*t + 1))**9, 45 * (t - 2)**8 / (2*t + 1)**10, var=t)
```

$$\frac{d}{dt}\left(\frac{(t-2)^9}{(2t+1)^9}\right) = \frac{45\left(t-2\right)^8}{\left(2t+1\right)^{10}}$$

```
Agrees with SymPy: True
```

Example 6. Differentiate $y = (2x+1)^5(x^3-x+1)^4$.

By hand. The Product Rule on the outside, the Chain Rule for each factor:

$$\begin{aligned} y' &= (2x+1)^5\cdot 4(x^3-x+1)^3(3x^2-1) + (x^3-x+1)^4\cdot 5(2x+1)^4\cdot 2 \\ &= 2(2x+1)^4(x^3-x+1)^3\big[\,2(2x+1)(3x^2-1) + 5(x^3-x+1)\,\big] \\ &= 2(2x+1)^4(x^3-x+1)^3\left(17x^3+6x^2-9x+3\right). \end{aligned}$$

Factoring out the common powers $(2x+1)^4(x^3-x+1)^3$ keeps the answer readable.

```
check((2*x + 1)**5 * (x**3 - x + 1)**4,
      2 * (2*x + 1)**4 * (x**3 - x + 1)**3 * (17*x**3 + 6*x**2 - 9*x + 3))
```

$$\frac{d}{dx}\left(\left(2x+1\right)^5\left(x^3-x+1\right)^4\right) = 2\left(2x+1\right)^4\left(x^3-x+1\right)^3\left(17x^3+6x^2-9x+3\right)$$

```
Agrees with SymPy: True
```

4. Exponential functions

Example 7. Differentiate $y = e^{\sin x}$.

By hand. The outer function is e^u (its own derivative), the inner function is $u = \sin x$:

$$\frac{dy}{dx} = e^{\sin x}\cdot\cos x.$$

In general, $\dfrac{d}{dx}\,e^{g(x)} = e^{g(x)}\,g'(x)$.

```
check(sp.exp(sp.sin(x)), sp.exp(sp.sin(x)) * sp.cos(x))
```

$$\frac{d}{dx}\left(e^{\sin\left(x\right)}\right) = e^{\sin\left(x\right)}\cos\left(x\right)$$

```
Agrees with SymPy: True
```

Derivative of b^x. For any base $b > 0$, write $b = e^{\ln b}$, so $b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$. The Chain Rule with inner function $(\ln b)\,x$ gives

$$\frac{d}{dx}\left(b^x\right) = e^{(\ln b)x}\cdot\ln b = b^x\ln b.$$

$$\boxed{\frac{d}{dx}\left(b^x\right) = b^x \ln b}$$

When $b = e$, $\ln e = 1$ and we recover $\dfrac{d}{dx}e^x = e^x$. This is why e is the "natural" base: it is the only base with no extra constant.

Example 8. $\dfrac{d}{dx}\left(2^x\right) = 2^x \ln 2 \approx (0.693)\, 2^x$. Compare §3.1, where we estimated $\lim_{h\to 0} \dfrac{2^h - 1}{h} \approx 0.69$.

```
b = sp.symbols("b", positive=True)
check(b**x, b**x * sp.log(b))
check(2**x, 2**x * sp.log(2))
print("ln 2  ", float(sp.log(2)))
```

$\dfrac{d}{dx}\left(b^x\right) = b^x \log\left(b\right)$

```
Agrees with SymPy: True
```

$\dfrac{d}{dx}\left(2^x\right) = 2^x \log\left(2\right)$

```
Agrees with SymPy: True
ln 2   0.6931471805599453
```

5. Longer chains

The Chain Rule extends to compositions of three or more functions. If $y = f(u)$, $u = g(v)$, $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx}, \qquad \text{i.e.} \qquad \frac{d}{dx} f\Big(g(h(x))\Big) = f'\Big(g(h(x))\Big) \cdot g'(h(x)) \cdot h'(x).$$

Work from the **outside in**, one layer at a time.

Example 9. If $f(x) = \sin(\cos(\tan x))$, then

$$\begin{aligned} f'(x) &= \cos(\cos(\tan x)) \cdot \frac{d}{dx}\cos(\tan x) \\ &= \cos(\cos(\tan x)) \cdot \big[-\sin(\tan x)\big] \cdot \frac{d}{dx}\tan x \\ &= -\cos(\cos(\tan x))\, \sin(\tan x)\, \sec^2 x. \end{aligned}$$

```
check(sp.sin(sp.cos(sp.tan(x))),
      -sp.cos(sp.cos(sp.tan(x))) * sp.sin(sp.tan(x)) * sp.sec(x)**2)
```

$\dfrac{d}{dx}\left(\sin\left(\cos\left(\tan\left(x\right)\right)\right)\right) = -\sin\left(\tan\left(x\right)\right)\cos\left(\cos\left(\tan\left(x\right)\right)\right)\sec^{2}\left(x\right)$

```
Agrees with SymPy: True
```

Example 10. Differentiate $y = e^{\sec 3\theta}$.

By hand. Three layers: $e^{(\cdot)}$, then $\sec(\cdot)$, then 3θ.

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \cdot \frac{d}{d\theta}(\sec 3\theta) = e^{\sec 3\theta} \cdot \sec 3\theta \tan 3\theta \cdot \frac{d}{d\theta}(3\theta) = 3\, e^{\sec 3\theta} \sec 3\theta \tan 3\theta.$$

```
check(sp.exp(sp.sec(3*theta)),
      3 * sp.exp(sp.sec(3*theta)) * sp.sec(3*theta) * sp.tan(3*theta), var=theta)
```

$\dfrac{d}{d\theta}\left(e^{\sec\left(3\theta\right)}\right) = 3e^{\sec\left(3\theta\right)}\tan\left(3\theta\right)\sec\left(3\theta\right)$

```
Agrees with SymPy: True
```

Example 11 (values from a table). Suppose $h(x) = f(g(x))$ with $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 5$. Then

$$h'(1) = f'(g(1)) \cdot g'(1) = f'(4) \cdot 3 = 15.$$

Note that we need f' at $g(1) = 4$, **not** at 1.

```
# h'(1) depends only on g(1), g'(1) and f'(4), so any f and g with those values work,
# e.g. the lines g(x) = 4 + 3(x - 1) and f(u) = 5u.
g_line = 4 + 3 * (x - 1)
f_line = lambda w: 5 * w
print("h'(1) =", sp.diff(f_line(g_line), x).subs(x, 1))
```

```
h'(1) = 15
```

6. A numerical sanity check

As in earlier sections, we compare each by-hand derivative with a **central difference** $\dfrac{f(x+h) - f(x-h)}{2h}$ at many points.

```
def numeric_check(f, fprime, a=0.2, b=1.2, step=1e-6):
    fn, dfn = sp.lambdify(x, f, "numpy"), sp.lambdify(x, fprime, "numpy")
    xs = np.linspace(a, b, 200)
    approx = (fn(xs + step) - fn(xs - step)) / (2 * step)
    exact = dfn(xs)
    return np.max(np.abs(approx - exact)) / np.max(np.abs(exact))

tests = {
    r"\sqrt{x^2+1}": (sp.sqrt(x**2 + 1), x / sp.sqrt(x**2 + 1)),
    r"\sin(x^2)": (sp.sin(x**2), 2*x * sp.cos(x**2)),
    r"(x^3-1)^{100}": ((x**3 - 1)**100, 300 * x**2 * (x**3 - 1)**99),
    r"(2x+1)^5(x^3-x+1)^4": ((2*x + 1)**5 * (x**3 - x + 1)**4,
                             2 * (2*x + 1)**4 * (x**3 - x + 1)**3 * (17*x**3 + 6*x**2 - 9*x + 3)),
    r"e^{\sin x}": (sp.exp(sp.sin(x)), sp.exp(sp.sin(x)) * sp.cos(x)),
    r"2^x": (2**x, 2**x * sp.log(2)),
    r"\sin(\cos(\tan x))": (sp.sin(sp.cos(sp.tan(x))),
                            -sp.cos(sp.cos(sp.tan(x))) * sp.sin(sp.tan(x)) * sp.sec(x)**2),
}
for name, (f, fp) in tests.items():
    rel = numeric_check(f, fp)
    display(Math(rf"{name}: \quad \text{{relative error}} \approx {rel:.1e}"))
```

$\sqrt{x^2+1}$: relative error $\approx 2.0e-10$

$\sin(x^2)$: relative error $\approx 1.2e-10$

$(x^3-1)^{100}$: relative error $\approx 2.9e-10$

$(2x+1)^5(x^3-x+1)^4$: relative error $\approx 6.1e-11$

$e^{\sin x}$: relative error $\approx 2.5e-10$

2^x: relative error $\approx 1.8e-10$

$\sin(\cos(\tan x))$: relative error $\approx 1.3e-10$

7. Why the Chain Rule is true

A first attempt. Let $\Delta u = g(x+\Delta x) - g(x)$ and $\Delta y = f(u+\Delta u) - f(u)$. It is tempting to write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and let $\Delta x \to 0$. Since g is continuous, $\Delta u \to 0$, so the two factors go to $f'(u)$ and $g'(x)$. **The problem:** Δu may equal 0 for $\Delta x \neq 0$ (for example, when g is constant near x), and then we divided by zero.

A correct proof. Since g is differentiable at a, we can write

$$\Delta u = g'(a)\,\Delta x + \varepsilon_1\,\Delta x, \qquad \text{where } \varepsilon_1 \to 0 \text{ as } \Delta x \to 0.$$

Similarly, with $b = g(a)$, since f is differentiable at b,

$$\Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u, \qquad \text{where } \varepsilon_2 \to 0 \text{ as } \Delta u \to 0$$

(define $\varepsilon_2 = 0$ when $\Delta u = 0$, so this holds even then). Substituting the first equation into the second,

$$\Delta y = [f'(b)+\varepsilon_2][g'(a)+\varepsilon_1]\Delta x \quad \Longrightarrow \quad \frac{\Delta y}{\Delta x} = [f'(b)+\varepsilon_2][g'(a)+\varepsilon_1].$$

No division by Δu is needed. As $\Delta x \to 0$, $\Delta u \to 0$, so $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(b)\,g'(a) = f'(g(a))\,g'(a). \qquad \blacksquare$$

8. Practice problems

Try these by hand first, then run the cell below to check.

1. $y = (5x^4 - 3x)^7$
2. $f(x) = \sqrt{1-2x}$
3. $y = \cos(a^3 + x^3)$, where a is a constant
4. $y = xe^{-x^2}$
5. $g(t) = \tan^2(3t)$
6. $y = 10^{\,1-x^2}$
7. $y = \sin(\,\sin(\sin x))$
8. Find the tangent line to $y = (1+2x)^{10}$ at $(0, 1)$.
9. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

```
a = sp.symbols("a")
problems = [
    ((5*x**4 - 3*x)**7, x),
    (sp.sqrt(1 - 2*x), x),
    (sp.cos(a**3 + x**3), x),
    (x * sp.exp(-x**2), x),
    (sp.tan(3*t)**2, t),
    (10**(1 - x**2), x),
    (sp.sin(sp.sin(sp.sin(x))), x),
]
for i, (f, var) in enumerate(problems, 1):
    df = sp.diff(f, var)
    df = sp.factor(df) if i in (1, 4) else df
    display(Math(rf"\text{{{i}.}}\quad \frac{{d}}{{d{sp.latex(var)}}}\left({sp.latex(f)}\right) = {sp.la

# 8. tangent line
y = (1 + 2*x)**10
```

```
m = sp.diff(y, x).subs(x, 0)
display(Math(rf"\text{{8.}}\quad y = {sp.latex(sp.expand(1 + m*x))}"))

# 9. F'(5) = f'(g(5)) g'(5) = f'(-2) g'(5)
display(Math(rf"\text{{9.}}\quad F'(5) = f'(-2)\cdot g'(5) = {4 * 6}"))
```

1. $\dfrac{d}{dx}\left(\left(5x^4 - 3x\right)^7\right) = 7x^6\left(5x^3 - 3\right)^6\left(20x^3 - 3\right)$
2. $\dfrac{d}{dx}\left(\sqrt{1-2x}\right) = -\dfrac{1}{\sqrt{1-2x}}$
3. $\dfrac{d}{dx}\left(\cos\left(a^3 + x^3\right)\right) = -3x^2\sin\left(a^3 + x^3\right)$
4. $\dfrac{d}{dx}\left(xe^{-x^2}\right) = -\left(2x^2 - 1\right)e^{-x^2}$
5. $\dfrac{d}{dt}\left(\tan^2(3t)\right) = \left(6\tan^2(3t) + 6\right)\tan(3t)$
6. $\dfrac{d}{dx}\left(10^{1-x^2}\right) = -2\cdot 10^{1-x^2}\,x\log(10)$
7. $\dfrac{d}{dx}\left(\sin\left(\sin\left(\sin(x)\right)\right)\right) = \cos(x)\cos\left(\sin(x)\right)\cos\left(\sin\left(\sin(x)\right)\right)$
8. $y = 20x + 1$
9. $F'(5) = f'(-2)\cdot g'(5) = 24$

Summary

Function	Derivative	Rule
$f(g(x))$	$f'(g(x))\,g'(x)$	Chain
$[g(x)]^n$	$n[g(x)]^{n-1}g'(x)$	Power + Chain
$e^{g(x)}$	$e^{g(x)}\,g'(x)$	Chain
b^x	$b^x \ln b$	Chain ($b^x = e^{x\ln b}$)

In Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Key ideas. - Identify the **outer** and **inner** functions. Differentiate the outer function, leave the inner function unchanged inside it, then multiply by the derivative of the inner function. - $\sin(x^2) \neq \sin^2 x$: they are different compositions with different derivatives. - Don't expand large powers like $(x^3-1)^{100}$; use the Power Rule with the Chain Rule. - Rewrite roots and reciprocals as powers to avoid the Quotient Rule. - For longer chains, work from the outside in, one layer at a time. - The naive "cancel du" proof fails when $\Delta u = 0$; the ε proof avoids dividing by Δu.

Previous: §3.3 Derivatives of Trigonometric Functions.

Next: §3.5 Implicit Differentiation.