Lecture Notes: Derivatives of Trigonometric Functions

Miraj Samarakkody

2026-10-29

Table of contents

1. Where we are . . . 1
2. A graphical guess . . . 2
3. Two important limits . . . 2
4. The derivatives of sine and cosine . . . 4
5. The other trigonometric functions . . . 4
6. Trigonometric limits . . . 8
7. A numerical sanity check . . . 8
8. Practice problems . . . 9
Summary . . . 10

*Stewart, **Calculus**, 9th ed., §3.3*

Learning objectives. By the end of this lecture you should be able to: 1. Explain the two key limits $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ and $\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$, and why they need **radians**. 2. Derive $\frac{d}{dx}(\sin x)=\cos x$ and $\frac{d}{dx}(\cos x)=-\sin x$ from the limit definition. 3. Use the Quotient Rule to find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$. 4. Apply these formulas to products, quotients, higher derivatives, and simple harmonic motion. 5. Evaluate trigonometric limits by reducing them to $\frac{\sin\theta}{\theta}$. 6. Check every answer symbolically with **SymPy** and numerically with **NumPy**.

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import Math, display

sp.init_printing()
x, t, h, theta = sp.symbols("x t h theta")
```

1. Where we are

From §3.1 and §3.2 we can differentiate:

Function	Derivative
c	0
x^n	nx^{n-1}
e^x	e^x
$c\,f \pm g$	$c\,f' \pm g'$
$f\,g$	$fg'+gf'$
$\frac{f}{g}$	$\frac{gf'-fg'}{g^2}$

In this section we add the six trigonometric functions. **Throughout, angles are measured in radians.**

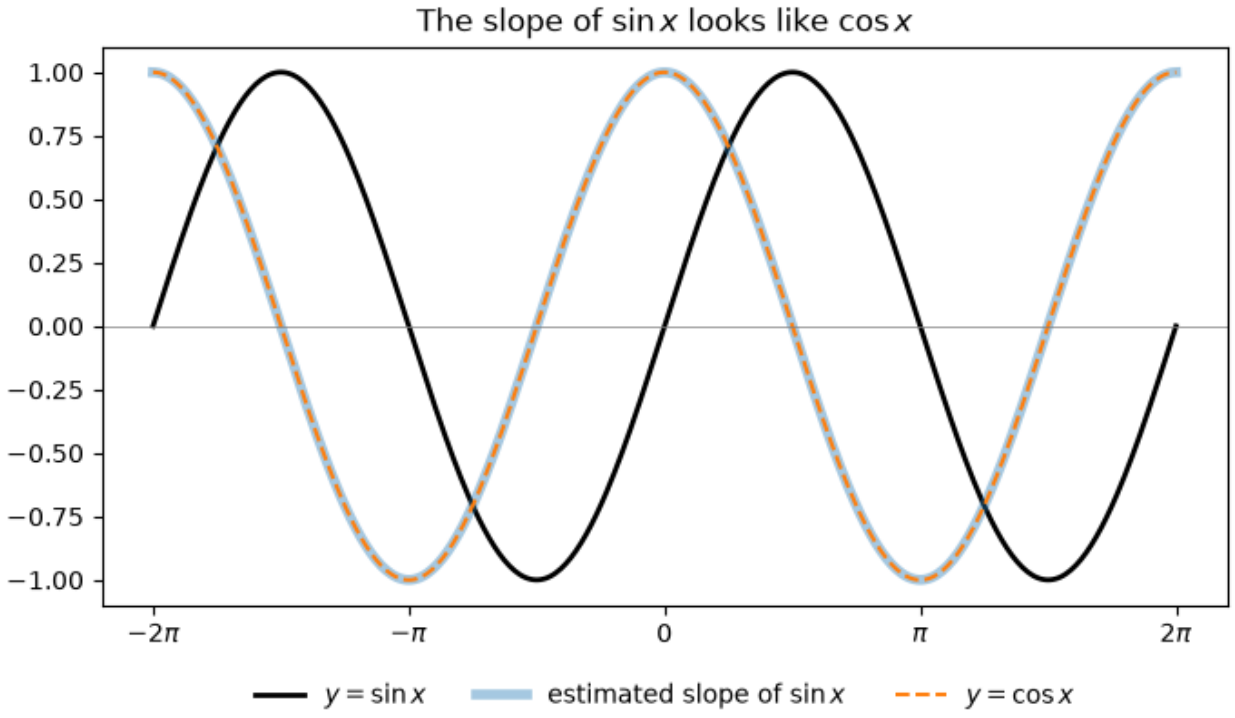
As before, the helper below compares a derivative found by hand with SymPy's answer.

2. A graphical guess

Before proving anything, let's estimate the slope of $y = \sin x$ at many points and plot it. The slope is 1 at $x = 0$, 0 at the peaks and troughs, and -1 at $x = \pi$. That pattern looks like $\cos x$.

```
xs = np.linspace(-2 * np.pi, 2 * np.pi, 400)
slopes = np.gradient(np.sin(xs), xs)  # numerical estimate of the slope

fig, ax = plt.subplots(figsize=(8, 4))
ax.plot(xs, np.sin(xs), "k", lw=2, label=r"$y = \sin x$")
ax.plot(xs, slopes, lw=4, alpha=0.4, label=r"estimated slope of $\sin x$")
ax.plot(xs, np.cos(xs), "--", lw=1.5, label=r"$y = \cos x$")
ax.axhline(0, color="gray", lw=0.5)
ax.set_xticks(np.arange(-2, 3) * np.pi, [r"$-2\pi$", r"$-\pi$", "0", r"$\pi$", r"$2\pi$"])
ax.legend(loc="upper center", bbox_to_anchor=(0.5, -0.1), ncol=3, frameon=False)
ax.set_title(r"The slope of $\sin x$ looks like $\cos x$")
plt.show()
```



3. Two important limits

To prove the guess, we need two limits.

Limit 1.

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Proof (squeeze). Take $0 < \theta < \pi/2$ and a unit circle. Comparing the areas of the small triangle, the circular

sector, and the large triangle that share the angle θ gives

$$\tfrac{1}{2}\sin\theta \;<\; \tfrac{1}{2}\theta \;<\; \tfrac{1}{2}\tan\theta.$$

Divide by $\tfrac{1}{2}\sin\theta > 0$ and take reciprocals:

$$\cos\theta \;<\; \frac{\sin\theta}{\theta} \;<\; 1.$$

As $\theta \to 0^+$, $\cos\theta \to 1$, so by the Squeeze Theorem $\dfrac{\sin\theta}{\theta} \to 1$. The function $\dfrac{\sin\theta}{\theta}$ is even, so the left-hand limit is also 1. ■

Limit 2.

$$\boxed{\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0}$$

Proof. Multiply the numerator and denominator by $\cos\theta + 1$ and use $\cos^2\theta - 1 = -\sin^2\theta$:

$$\frac{\cos\theta - 1}{\theta} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = -\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta + 1} \xrightarrow{\theta\to 0} -1\cdot\frac{0}{2} = 0. \qquad \blacksquare$$

A table of values supports both limits:

```
print(f"{'theta':>10} {'sin( )/ ':>14} {'(cos( )-1)/ ':>16}")
for th in [1, 0.5, 0.1, 0.01, 0.001]:
    print(f"{th:>10} {np.sin(th) / th:>14.10f} {(np.cos(th) - 1) / th:>16.10f}")

print()
print("SymPy: lim sin( )/     =", sp.limit(sp.sin(theta) / theta, theta, 0))
print("SymPy: lim (cos( )-1)/ =", sp.limit((sp.cos(theta) - 1) / theta, theta, 0))
```

```
     theta       sin( )/     (cos( )-1)/
         1   0.8414709848    -0.4596976941
       0.5   0.9588510772    -0.2448348762
       0.1   0.9983341665    -0.0499583472
      0.01   0.9999833334    -0.0049999583
     0.001   0.9999998333    -0.0005000000

SymPy: lim sin( )/     = 1
SymPy: lim (cos( )-1)/ = 0
```

Why radians? The proof used the fact that a sector of angle θ in a unit circle has area $\tfrac{1}{2}\theta$, which is true only when θ is in radians. If θ is in degrees, then $\sin(\theta^\circ) = \sin\!\left(\frac{\pi\theta}{180}\right)$ and

$$\lim_{\theta\to 0}\frac{\sin(\theta^\circ)}{\theta} = \frac{\pi}{180} \approx 0.01745.$$

Every derivative formula below would pick up an ugly factor of $\pi/180$.

```
print("lim sin( °)/ =", sp.limit(sp.sin(sp.pi * theta / 180) / theta, theta, 0),
      " ", float(sp.pi / 180))
```

```
lim sin( °)/ = pi/180   0.017453292519943295
```

4. The derivatives of sine and cosine

Derivative of sine.

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Proof. Use the addition formula $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \xrightarrow{h \to 0} \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \qquad \blacksquare$$

Derivative of cosine.

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Proof. Use $\cos(x+h) = \cos x \cos h - \sin x \sin h$:

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \xrightarrow{h \to 0} \cos x \cdot 0 - \sin x \cdot 1 = -\sin x. \qquad \blacksquare$$

SymPy can take the limit of the difference quotient directly:

```
for f in [sp.sin(x), sp.cos(x)]:
    dq = (f.subs(x, x + h) - f) / h
    display(Math(rf"\lim_{{h\to0}} {sp.latex(dq)} = {sp.latex(sp.limit(dq, h, 0))}"))
```

$$\lim_{h \to 0} \frac{-\sin(x) + \sin(h+x)}{h} = \cos(x)$$

$$\lim_{h \to 0} \frac{-\cos(x) + \cos(h+x)}{h} = -\sin(x)$$

Example 1. Differentiate $y = x^2 \sin x$.

By hand (Product Rule).

$$y' = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) = x^2 \cos x + 2x \sin x.$$

```
y_prime = sp.diff(x**2 * sp.sin(x), x)
y_prime
```

$x^2 \cos(x) + 2x \sin(x)$

5. The other trigonometric functions

The remaining four functions are quotients of sin and cos, so the Quotient Rule does the work.

Tangent.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Secant.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x.$$

The derivatives of $\cot x$ and $\csc x$ follow the same way (try them!).

Derivatives of trigonometric functions.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$

Memory aid: the functions whose names start with **"c"** (cosine, cosecant, cotangent) have a **minus sign** in their derivatives. Each "co-" formula comes from the matching formula by swapping every function with its co-function and adding a minus sign.

Example 2. Differentiate $f(x) = \dfrac{\sec x}{1+\tan x}$. For which values of x does the graph of f have a horizontal tangent?

By hand (Quotient Rule).

$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2} = \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \frac{\sec x\,(\tan x - 1)}{(1+\tan x)^2},$$

using $\sec^2 x = \tan^2 x + 1$.

Since $\sec x$ is never 0, $f'(x) = 0$ exactly when $\tan x = 1$, that is, at $x = n\pi + \dfrac{\pi}{4}$ for any integer n.

```
f = sp.sec(x) / (1 + sp.tan(x))
df = sp.diff(f, x)

# the derivative, raw and simplified
display(Math(rf"f'(x) = {sp.latex(df)}"))
display(Math(rf"f'(x) = {sp.latex(sp.factor(sp.simplify(df)))}"))

# horizontal tangents on [0, 2 )
roots = sp.solveset(sp.tan(x) - 1, x, sp.Interval.Ropen(0, 2 * sp.pi))
display(Math(rf"\tan x = 1 \text{{ on }} [0, 2\pi): \quad x \in {sp.latex(roots)}"))

values = [sp.simplify(df.subs(x, r)) for r in roots]
display(Math(", \\quad ".join(rf"f'\left({sp.latex(r)}\right) = {sp.latex(v)}"
                              for r, v in zip(roots, values))))
```

$$f'(x) = \frac{\tan(x)\sec(x)}{\tan(x)+1} + \frac{(-\tan^2(x)-1)\sec(x)}{(\tan(x)+1)^2}$$

$$f'(x) = \frac{(\tan(x)-1)\sec(x)}{(\tan(x)+1)^2}$$

$$\tan x = 1 \text{ on } [0, 2\pi): \quad x \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$$

$$f'\left(\frac{\pi}{4}\right) = 0, \quad f'\left(\frac{5\pi}{4}\right) = 0$$

Example 3 (simple harmonic motion). An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Its position at time t is

$$s = f(t) = 4\cos t.$$

Find the velocity and acceleration at time t, and use them to describe the motion.

By hand.

$$v = \frac{ds}{dt} = -4\sin t, \qquad a = \frac{dv}{dt} = -4\cos t.$$

Interpretation. - The object moves between $s = 4$ and $s = -4$, with period 2π. - The **speed** $|v| = 4|\sin t|$ is largest when $\cos t = 0$, that is, as the object passes through its rest position $s = 0$. - $a = -s$: the acceleration always points back toward the rest position and is largest at the ends of the motion, where the spring is stretched or compressed the most.

```
for f in [sp.sin(x), sp.cos(x), sp.tan(x), sp.csc(x), sp.sec(x), sp.cot(x)]:
    display(Math(rf"\frac{{d}}{{dx}}{sp.latex(f)} = {sp.latex(sp.diff(f, x))}"))


ts = np.linspace(0, 2 * np.pi, 300)
fig, ax = plt.subplots(figsize=(8, 4))
ax.plot(ts, 4 * np.cos(ts), "k", lw=2, label=r"position $s = 4\cos t$")
ax.plot(ts, -4 * np.sin(ts), lw=1.5, label=r"velocity $v = -4\sin t$")
ax.plot(ts, -4 * np.cos(ts), "--", lw=1.5, label=r"acceleration $a = -4\cos t$")
ax.axhline(0, color="gray", lw=0.5)
ax.set_xticks(np.arange(5) * np.pi / 2, ["0", r"$\pi/2$", r"$\pi$", r"$3\pi/2$", r"$2\pi$"])
ax.set_xlabel("$t$")
ax.legend(loc="upper center", bbox_to_anchor=(0.5, -0.18), ncol=3, frameon=False)
ax.set_title("A mass on a spring")
plt.show()
```

$$\frac{d}{dx}\sin\left(x\right) = \cos\left(x\right)$$

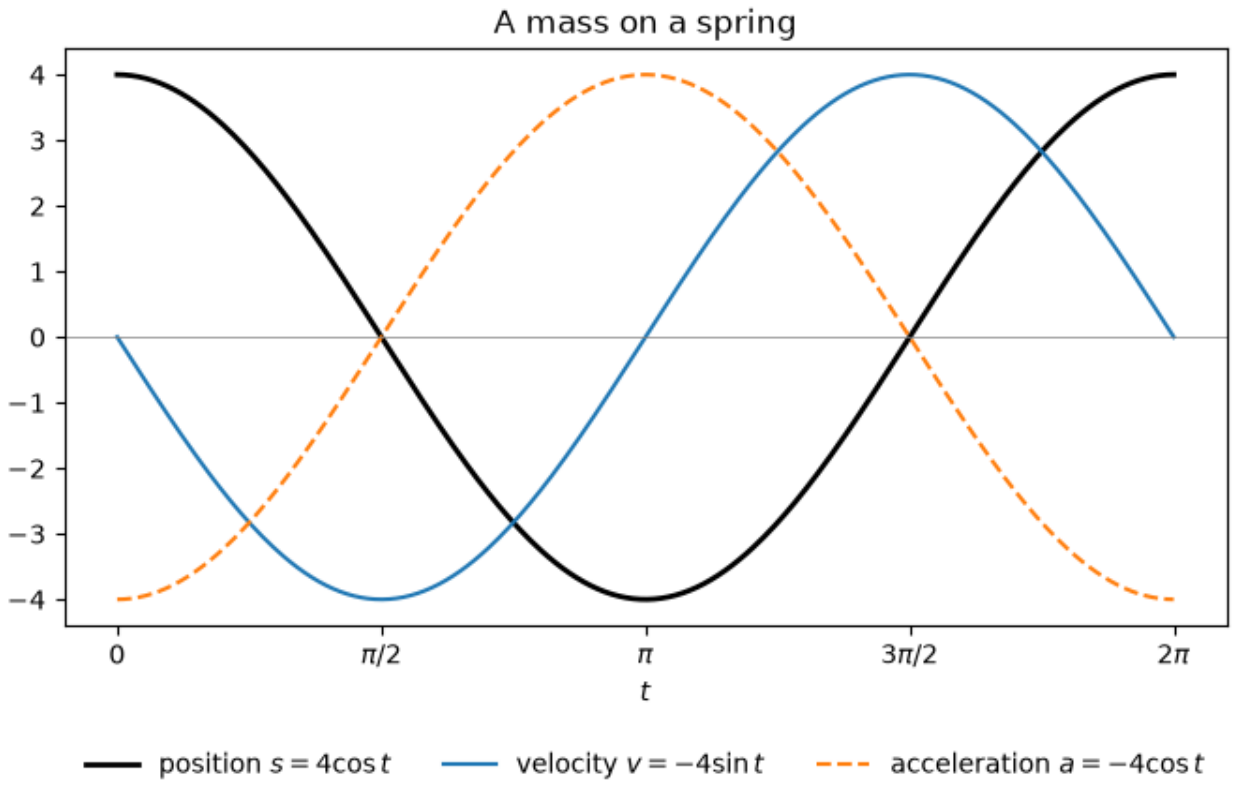
$$\frac{d}{dx}\cos\left(x\right) = -\sin\left(x\right)$$

$$\frac{d}{dx}\tan\left(x\right) = \tan^{2}\left(x\right) + 1$$

$$\frac{d}{dx}\csc\left(x\right) = -\cot\left(x\right)\csc\left(x\right)$$

$$\frac{d}{dx}\sec\left(x\right) = \tan\left(x\right)\sec\left(x\right)$$

$$\frac{d}{dx}\cot\left(x\right) = -\cot^{2}\left(x\right) - 1$$



Example 4. Find the 27th derivative of $\cos x$.

By hand. The derivatives of $\cos x$ repeat in a cycle of four:

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x, \ \ldots$$

So $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Since $27 = 4 \cdot 6 + 3$,

$$\frac{d^{27}}{dx^{27}}(\cos x) = \frac{d^3}{dx^3}(\cos x) = \sin x.$$

```
for k in range(9):
    display(Math(rf"f^{{({k})}}(x) = {sp.latex(sp.diff(sp.cos(x), x, k))}"))
display(Math(rf"f^{{(27)}}(x) = {sp.latex(sp.diff(sp.cos(x), x, 27))}"))
```

$f^{(0)}(x) = \cos\left(x\right)$

$f^{(1)}(x) = -\sin\left(x\right)$

$f^{(2)}(x) = -\cos\left(x\right)$

$f^{(3)}(x) = \sin\left(x\right)$

$f^{(4)}(x) = \cos\left(x\right)$

$f^{(5)}(x) = -\sin\left(x\right)$

$f^{(6)}(x) = -\cos\left(x\right)$

$f^{(7)}(x) = \sin\left(x\right)$

$f^{(8)}(x) = \cos\left(x\right)$

$f^{(27)}(x) = \sin\left(x\right)$

6. Trigonometric limits

Limit 1 is useful well beyond proving derivative formulas. The trick is to make the argument of sin match the denominator.

Example 5. Find $\lim_{x\to 0} \dfrac{\sin 7x}{4x}$.

By hand. Multiply and divide by 7, and let $\theta = 7x$ (so $\theta \to 0$ as $x \to 0$):

$$\lim_{x\to 0} \frac{\sin 7x}{4x} = \frac{7}{4} \lim_{x\to 0} \frac{\sin 7x}{7x} = \frac{7}{4} \lim_{\theta\to 0} \frac{\sin \theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}.$$

Note: $\sin 7x \neq 7 \sin x$, so we cannot just cancel the 7.

Example 6. Find $\lim_{x\to 0} x \cot x$.

By hand. Write $\cot x = \dfrac{\cos x}{\sin x}$ and divide the numerator and denominator by x:

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \dfrac{\sin x}{x}} = \frac{1}{1} = 1.$$

```
for expr, by_hand in [(sp.sin(7*x) / (4*x), sp.Rational(7, 4)), (x * sp.cot(x), 1)]:
    lim = sp.limit(expr, x, 0)
    display(Math(rf"\lim_{{x\to0}} {sp.latex(expr)} = {sp.latex(lim)}"))
    print("Agrees with by-hand answer:", lim == by_hand)
    fn = sp.lambdify(x, expr, "numpy")
    print("  numeric:", ", ".join(f"x={xv:g}: {fn(xv):.6f}" for xv in [0.1, 0.01, 0.001]))
```

$$\lim_{x\to 0} \frac{\sin\left(7x\right)}{4x} = \frac{7}{4}$$

```
Agrees with by-hand answer: True
  numeric: x=0.1: 1.610544, x=0.01: 1.748571, x=0.001: 1.749986
```

$$\lim_{x\to 0} x \cot\left(x\right) = 1$$

```
Agrees with by-hand answer: True
  numeric: x=0.1: 0.996664, x=0.01: 0.999967, x=0.001: 1.000000
```

7. A numerical sanity check

As in §3.2, we compare each by-hand derivative with a **central difference** $\dfrac{f(x+h) - f(x-h)}{2h}$ at many points. We use the interval $[0.2, 1.3]$, which avoids the vertical asymptotes of $\tan x$ and $\sec x$ at $x = \pi/2 \approx 1.57$.

```
def numeric_check(f, fprime, a=0.2, b=1.3, step=1e-5):
    fn, dfn = sp.lambdify(x, f, "numpy"), sp.lambdify(x, fprime, "numpy")
    xs = np.linspace(a, b, 200)
    approx = (fn(xs + step) - fn(xs - step)) / (2 * step)
    exact = dfn(xs)
    return np.max(np.abs(approx - exact)) / np.max(np.abs(exact))

tests = {
    r"\sin x": (sp.sin(x), sp.cos(x)),
    r"\cos x": (sp.cos(x), -sp.sin(x)),
    r"\tan x": (sp.tan(x), sp.sec(x)**2),
    r"\csc x": (sp.csc(x), -sp.csc(x) * sp.cot(x)),
    r"\sec x": (sp.sec(x), sp.sec(x) * sp.tan(x)),
```

```
    r"\cot x": (sp.cot(x), -sp.csc(x)**2),
    r"x^2 \sin x": (x**2 * sp.sin(x), x**2 * sp.cos(x) + 2*x * sp.sin(x)),
    r"\frac{\sec x}{1 + \tan x}": (sp.sec(x) / (1 + sp.tan(x)),
                                   sp.sec(x) * (sp.tan(x) - 1) / (1 + sp.tan(x))**2),
}
for name, (f, fp) in tests.items():
    rel = numeric_check(f, fp)
    display(Math(rf"{name}: \quad \text{{relative error}} \approx {rel:.1e}"))
```

$\sin x$: relative error $\approx 2.2e-11$

$\cos x$: relative error $\approx 2.3e-11$

$\tan x$: relative error $\approx 1.3e-09$

$\csc x$: relative error $\approx 2.5e-09$

$\sec x$: relative error $\approx 1.4e-09$

$\cot x$: relative error $\approx 2.5e-09$

$x^2 \sin x$: relative error $\approx 3.2e-11$

$\dfrac{\sec x}{1+\tan x}$: relative error $\approx 1.3e-10$

8. Practice problems

Try these by hand first, then run the cell below to check.

1. $f(x) = 3\sin x - 2\cos x$
2. $y = e^x \cos x$
3. $y = \dfrac{\sin x}{1+\cos x}$ (*simplify your answer as far as you can*)
4. $f(\theta) = \theta \cos\theta \sin\theta$
5. Find the tangent line to $y = \sec x$ at $\left(\dfrac{\pi}{3}, 2\right)$.
6. Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$ and $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x}$.
7. Find $\dfrac{d^{99}}{dx^{99}}(\sin x)$.

```
problems = [
    (3*sp.sin(x) - 2*sp.cos(x), x),
    (sp.exp(x) * sp.cos(x), x),
    (sp.sin(x) / (1 + sp.cos(x)), x),
    (theta * sp.cos(theta) * sp.sin(theta), theta),
]
for i, (f, var) in enumerate(problems, 1):
    df = sp.diff(f, var)
    df = sp.trigsimp(sp.simplify(df)) if i == 3 else df
    display(Math(rf"\text{{{i}.}}\quad \frac{{d}}{{d{sp.latex(var)}}}\left({sp.latex(f)}\right) = {sp.la

# 5. tangent line
y = sp.sec(x)
m = sp.diff(y, x).subs(x, sp.pi / 3)
display(Math(rf"\text{{5.}}\quad y = 2 + {sp.latex(m)}\left(x - \frac{{\pi}}{{3}}\right)"))

# 6. limits
l1 = sp.limit(sp.sin(3*x) / sp.sin(5*x), x, 0)
```

```
l2 = sp.limit(sp.sin(x**2) / x, x, 0)
display(Math(rf"\text{{6.}}\quad {sp.latex(l1)}, \qquad {sp.latex(l2)}"))

# 7. 99 = 4·24 + 3
display(Math(rf"\text{{7.}}\quad {sp.latex(sp.diff(sp.sin(x), x, 99))}"))
```

1. $\dfrac{d}{dx}\left(3\sin\left(x\right) - 2\cos\left(x\right)\right) = 2\sin\left(x\right) + 3\cos\left(x\right)$
2. $\dfrac{d}{dx}\left(e^{x}\cos\left(x\right)\right) = -e^{x}\sin\left(x\right) + e^{x}\cos\left(x\right)$
3. $\dfrac{d}{dx}\left(\dfrac{\sin\left(x\right)}{\cos\left(x\right)+1}\right) = \dfrac{1}{\cos\left(x\right)+1}$
4. $\dfrac{d}{d\theta}\left(\theta\sin\left(\theta\right)\cos\left(\theta\right)\right) = -\theta\sin^{2}\left(\theta\right) + \theta\cos^{2}\left(\theta\right) + \sin\left(\theta\right)\cos\left(\theta\right)$
5. $y = 2 + 2\sqrt{3}\left(x - \dfrac{\pi}{3}\right)$
6. $\dfrac{3}{5}, \qquad 0$
7. $-\cos\left(x\right)$

Summary

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin x$	$\cos x$	$\csc x$	$-\csc x\cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x\tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$

Two key limits (with θ in radians):

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad\qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

Key ideas. - The derivative formulas hold only when angles are measured in **radians**. - sin and cos come from the limit definition and the addition formulas; the other four come from the Quotient Rule. - The "co-" functions have minus signs in their derivatives. - Derivatives of $\sin x$ and $\cos x$ repeat every four steps, so reduce n modulo 4 for high derivatives. - For limits like $\dfrac{\sin kx}{x}$, rewrite them so the argument of sin matches the denominator.

Previous: §3.2 The Product and Quotient Rules.

Next: §3.4 The Chain Rule.