Techniques of Integration — Rules and Formulas

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2026-10-26

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Based on Stewart, ***Calculus****, Chapter 7: Techniques of Integration*

Throughout, C is an arbitrary constant and $a \neq 0$.

Table of Integration Formulas (Stewart, §7.5)

#	Integral
1	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
2	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
3	$\int e^x\,dx = e^x + C$
4	$\int b^x\,dx = \dfrac{b^x}{\ln b} + C$
5	$\int \sin x\,dx = -\cos x + C$
6	$\int \cos x\,dx = \sin x + C$
7	$\int \sec^2 x\,dx = \tan x + C$
8	$\int \csc^2 x\,dx = -\cot x + C$

#	Integral
9	$\int \sec x \tan x \, dx = \sec x + C$
10	$\int \csc x \cot x \, dx = -\csc x + C$
11	$\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$
12	$\int \csc x \, dx = \ln\lvert\csc x - \cot x\rvert + C$
13	$\int \tan x \, dx = \ln\lvert\sec x\rvert + C$
14	$\int \cot x \, dx = \ln\lvert\sin x\rvert + C$
15	$\int \sinh x \, dx = \cosh x + C$
16	$\int \cosh x \, dx = \sinh x + C$
17	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
18	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, \ a > 0$
19	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left\lvert\frac{x - a}{x + a}\right\rvert + C$
20	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + C$

7.1 Integration by Parts

Derived from the Product Rule $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$.

Indefinite form

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx$$

Shorthand (with $u = f(x)$, $v = g(x)$):

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Definite form

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x)\Big|_a^b - \int_a^b g(x)\, f'(x)\, dx$$

Choosing u (LIATE heuristic): pick u as the first type in the list **L**ogarithmic → **I**nverse trig → **A**lgebraic → **T**rigonometric → **E**xponential, and let dv be the rest. Goal: $\int v\, du$ should be simpler than the original.

Tips - Repeated parts: e.g. $\int x^2 e^x \, dx$ — apply twice. - "Cyclic" integrals such as $\int e^x \sin x \, dx$: apply parts twice, then solve algebraically for the original integral. - Reduction formulas, e.g.

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \, \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Example: $\int x \sin x \, dx$, $u = x$, $dv = \sin x \, dx \Rightarrow du = dx$, $v = -\cos x$:

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

7.2 Trigonometric Integrals

Useful identities

$$\sin^2 x + \cos^2 x = 1, \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin x \cos x = \tfrac{1}{2}\sin 2x$$

Strategy for $\int \sin^m x \, \cos^n x \, dx$

Case	Strategy
(a) power of **cosine is odd** ($n = 2k+1$)	Save one $\cos x$, write $\cos^{2k} x = (1 - \sin^2 x)^k$, substitute $u = \sin x$
(b) power of **sine is odd** ($m = 2k+1$)	Save one $\sin x$, write $\sin^{2k} x = (1 - \cos^2 x)^k$, substitute $u = \cos x$
(c) **both even**	Use half-angle identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(If both are odd, either (a) or (b) works.)

Strategy for $\int \tan^m x \, \sec^n x \, dx$

Case	Strategy
(a) power of **secant is even** ($n = 2k,\ k \geq 2$)	Save $\sec^2 x$, write the rest via $\sec^2 x = 1 + \tan^2 x$, substitute $u = \tan x$
(b) power of **tangent is odd** ($m = 2k+1$)	Save $\sec x \tan x$, write the rest via $\tan^2 x = \sec^2 x - 1$, substitute $u = \sec x$
Other cases	Use identities, integration by parts, and $\int \tan x \, dx$, $\int \sec x \, dx$

Useful results:

$$\int \tan x \, dx = \ln|\sec x| + C, \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

(Integrals of $\cot^m x \, \csc^n x$ are handled the same way using $1 + \cot^2 x = \csc^2 x$.)

Products of sines and cosines with different angles

For $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$ use:

$$\sin A \cos B = \tfrac{1}{2}\left[\sin(A - B) + \sin(A + B)\right]$$

$$\sin A \sin B = \tfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\cos A \cos B = \tfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

7.3 Trigonometric Substitution

Used for integrands containing $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ (with $a>0$).

Expression	Substitution	Identity used	Restriction on θ
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

Steps 1. Substitute x and dx (e.g. $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$). 2. Simplify the radical with the identity, e.g. $\sqrt{a^2-a^2\sin^2\theta} = a\cos\theta$. 3. Integrate in θ. 4. Return to x using a **reference right triangle** (e.g. $\sin\theta = x/a$ gives opposite x, hypotenuse a, adjacent $\sqrt{a^2-x^2}$). 5. For definite integrals, change the limits to θ-values instead.

Tip: for $\sqrt{ax^2+bx+c}$, first **complete the square**, then use a trig substitution.

7.4 Integration of Rational Functions by Partial Fractions

To integrate $f(x) = \dfrac{P(x)}{Q(x)}$:

Step 0 — Improper fraction: if $\deg P \ge \deg Q$, use long division:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \qquad \deg R < \deg Q$$

Step 1 — Factor $Q(x)$ into linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c) with $b^2-4ac<0$.

Step 2 — Decompose according to the cases:

Case	Factor in $Q(x)$	Partial fraction terms
I	distinct linear factors $(a_1x+b_1)\cdots(a_kx+b_k)$	$\dfrac{A_1}{a_1x+b_1} + \cdots + \dfrac{A_k}{a_kx+b_k}$
II	repeated linear factor $(ax+b)^r$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_r}{(ax+b)^r}$
III	distinct irreducible quadratic ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
IV	repeated irreducible quadratic $(ax^2+bx+c)^r$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \cdots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

Step 3 — Solve for the constants (multiply through by $Q(x)$; equate coefficients or plug in convenient x-values).

Step 4 — Integrate each term. Key formulas:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C, \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\!\left(\frac{x}{a}\right) + C$$

Rationalizing substitutions: if the integrand contains $\sqrt[n]{g(x)}$, try $u = \sqrt[n]{g(x)}$ to obtain a rational function.

7.5 Strategy for Integration

1. **Simplify the integrand** if possible (algebra or trig identities). e.g. $\int \sqrt{x}(1+\sqrt{x})\,dx = \int(\sqrt{x}+x)\,dx$, $\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \sin\theta\cos\theta\,d\theta$.
2. **Look for an obvious substitution** $u = g(x)$ where $du = g'(x)\,dx$ also appears (up to a constant).
3. **Classify the integrand by form:**
   - (a) Trigonometric functions → §7.2 strategies
   - (b) Rational functions → partial fractions (§7.4)
   - (c) Products of different kinds of functions → integration by parts (§7.1)
   - (d) Radicals:
     - $\sqrt{\pm x^2 \pm a^2}$ → trig substitution (§7.3)
     - $\sqrt[n]{ax+b}$ → rationalizing substitution $u = \sqrt[n]{ax+b}$
4. **Try again:**
   - (a) Try a different substitution.
   - (b) Try integration by parts again (e.g. single functions like $\tan^{-1} x$, $\ln x$).
   - (c) Manipulate the integrand: rationalize, multiply by a clever form of 1, use identities.
   - (d) Relate to previous problems.
   - (e) Use several methods together.

Note: some continuous functions have no elementary antiderivative, e.g.

$$\int e^{x^2}dx, \quad \int \frac{e^x}{x}dx, \quad \int \sin(x^2)\,dx, \quad \int \frac{1}{\ln x}dx, \quad \int \sqrt{x^3+1}\,dx, \quad \int \frac{\sin x}{x}dx$$

7.6 Integration Using Tables and Computer Algebra Systems

- Match the integrand to an entry in a table of integrals (Stewart's Reference Pages), possibly after a substitution or completing the square.
- A CAS (e.g. SymPy, Mathematica) may give an answer in a different but equivalent form; it may omit absolute values or the constant C.

7.7 Approximate Integration

Divide $[a,b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n}$, with $x_i = a + i\,\Delta x$ and midpoints $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$.

Left / Right endpoint rules

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x, \qquad R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's Rule (n even)

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Also: $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$.

Error bounds If $|f''(x)| \le K$ on $[a,b]$:

$$|E_T| \le \frac{K(b-a)^3}{12n^2}, \qquad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

If $|f^{(4)}(x)| \le K$ on $[a,b]$:

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

7.8 Improper Integrals

Type 1: Infinite intervals

(a) If $\int_a^t f(x)\,dx$ exists for every $t \ge a$:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(b) If $\int_t^b f(x)\,dx$ exists for every $t \le b$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

The integral is **convergent** if the limit exists (finite) and **divergent** otherwise.

(c) If both $\int_{-\infty}^a f\,dx$ and $\int_a^\infty f\,dx$ converge (for any a):

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

p-integral:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \le 1$$

Type 2: Discontinuous integrands

(a) If f is continuous on $[a,b)$ and discontinuous at b:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

(b) If f is continuous on $(a,b]$ and discontinuous at a:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

(c) If f has a discontinuity at c, $a < c < b$, and both pieces converge:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Also: $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges if $p \ge 1$.

Comparison Test

Suppose f and g are continuous with $f(x) \ge g(x) \ge 0$ for $x \ge a$. - (a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent. - (b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.