Lecture Notes: Trigonometric Integrals

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*Stewart, **Calculus**, §7.2*

Learning objectives. By the end of this lecture you should be able to: 1. Integrate $\sin^m x \cos^n x$ by saving one factor and substituting, or with half-angle identities. 2. Integrate $\tan^m x \sec^n x$ the same way. 3. Find $\int \tan x\,dx$, $\int \sec x\,dx$ and $\int \sec^3 x\,dx$. 4. Use product-to-sum identities for $\sin mx \cos nx$, $\sin mx \sin nx$ and $\cos mx \cos nx$. 5. Check every answer symbolically with **SymPy** and numerically with **NumPy**.

```
import numpy as np
import sympy as sp
```

```
import matplotlib.pyplot as plt
from IPython.display import Math, display

sp.init_printing()
x, u = sp.symbols("x u")
```

1. The identities we need

Every technique in this section uses one of these identities to rewrite the integrand.

Name	Identity
Pythagorean	$\sin^2 x + \cos^2 x = 1$
Pythagorean (tan/sec)	$\tan^2 x + 1 = \sec^2 x$
Half-angle	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
Half-angle	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
Double-angle	$\sin x \cos x = \frac{1}{2}\sin 2x$

We also need these derivatives, because they decide which substitution works:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x.$$

Let's confirm the identities with SymPy:

```
identities = {
    r"\sin^2 x + \cos^2 x = 1": sp.sin(x)**2 + sp.cos(x)**2 - 1,
    r"\tan^2 x + 1 = \sec^2 x": sp.tan(x)**2 + 1 - sp.sec(x)**2,
    r"\sin^2 x = \tfrac12(1-\cos 2x)": sp.sin(x)**2 - (1 - sp.cos(2 * x)) / 2,
    r"\cos^2 x = \tfrac12(1+\cos 2x)": sp.cos(x)**2 - (1 + sp.cos(2 * x)) / 2,
    r"\sin x\cos x = \tfrac12\sin 2x": sp.sin(x) * sp.cos(x) - sp.sin(2 * x) / 2,
}
for name, diff in identities.items():
    display(Math(name + rf"\quad\text{{: {sp.simplify(diff) == 0}}}"))
```

$\sin^2 x + \cos^2 x = 1$: True

$\tan^2 x + 1 = \sec^2 x$: True

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$: True

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$: True

$\sin x \cos x = \frac{1}{2}\sin 2x$: True

A checking function

As in §7.1, we check every antiderivative by differentiating it. Trig answers can look very different and still be equal, so if `simplify` cannot reduce $F' - f$ to 0, the function also compares F' and f numerically at a few points.

```
def check(F, integrand, var=x):
    # Verify F' = integrand: symbolically first, then numerically at sample points
    diff = sp.diff(F, var) - integrand
    ok = sp.simplify(diff) == 0
    if not ok:
        pts = [0.3, 0.7, 1.1, 2.0, -0.4]
```

```
        ok = all(abs(complex(diff.subs(var, p).evalf())) < 1e-10 for p in pts)
    print("d/dx(answer) == integrand:", ok)
    return ok
```

2. Powers of sine and cosine: $\int \sin^m x \cos^n x \, dx$

Example 1: $\int \cos^3 x \, dx$

The substitution $u = \cos x$ doesn't work, because $du = -\sin x \, dx$ and there is no $\sin x$ in the integrand. Instead, **save one factor** of $\cos x$ for du and turn the rest into sines:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x.$$

Now $u = \sin x$, $du = \cos x \, dx$:

$$\int \cos^3 x \, dx = \int (1 - u^2) \, du = u - \tfrac{1}{3}u^3 + C = \sin x - \tfrac{1}{3}\sin^3 x + C.$$

```
integrand_u = 1 - u**2
G = sp.integrate(integrand_u, u)
F = G.subs(u, sp.sin(x))
display(Math(r"\int(1-u^2)\,du = " + sp.latex(G) + r"\qquad\Longrightarrow\qquad \int\cos^3x\,dx = " + s
check(F, sp.cos(x)**3);
```

$$\int (1 - u^2) \, du = -\frac{u^3}{3} + u \qquad \Longrightarrow \qquad \int \cos^3 x \, dx = -\frac{\sin^3(x)}{3} + \sin(x) + C$$

```
d/dx(answer) == integrand: True
```

Example 2: $\int \sin^5 x \cos^2 x \, dx$

This time the power of **sine** is odd. Save one $\sin x$ for du and convert the remaining $\sin^4 x$ into cosines:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \cdot \sin x = (1 - \cos^2 x)^2 \cos^2 x \, \sin x.$$

With $u = \cos x$, $du = -\sin x \, dx$:

$$\begin{aligned}\int \sin^5 x \cos^2 x \, dx &= -\int (1 - u^2)^2 u^2 \, du = -\int (u^2 - 2u^4 + u^6) \, du \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C.\end{aligned}$$

```
integrand_u = -(1 - u**2)**2 * u**2
G = sp.integrate(sp.expand(integrand_u), u)
F = G.subs(u, sp.cos(x))
display(Math(r"\int\sin^5x\cos^2x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.sin(x)**5 * sp.cos(x)**2);
```

$$\int \sin^5 x \cos^2 x \, dx = -\frac{\cos^7(x)}{7} + \frac{2\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C$$

```
d/dx(answer) == integrand: True
```

The strategy when one power is odd

Case	Save	Convert the rest with	Substitute
power of cos is **odd**	one $\cos x$	$\cos^2 x = 1 - \sin^2 x$	$u = \sin x$
power of sin is **odd**	one $\sin x$	$\sin^2 x = 1 - \cos^2 x$	$u = \cos x$

If both powers are odd, either choice works.

The function below does these steps for any $m, n \geq 0$ with at least one odd. It shows the integral in u, so you can compare it with your own work.

```
def sin_cos_odd(m, n):
    #   sin^m x cos^n x dx when m or n is odd, by saving one factor and substituting
    if n % 2 == 1:                          # save cos x, u = sin x, du = cos x dx
        sub, g = sp.sin(x), u**m * (1 - u**2)**((n - 1) // 2)
    elif m % 2 == 1:                        # save sin x, u = cos x, du = -sin x dx
        sub, g = sp.cos(x), -(1 - u**2)**((m - 1) // 2) * u**n
    else:
        raise ValueError("both powers are even: use the half-angle identities")
    G = sp.integrate(sp.expand(g), u)
    F = G.subs(u, sub)
    display(Math(
        rf"u = {sp.latex(sub)}:\quad \int \sin^{{{m}}}x\cos^{{{n}}}x\,dx = \int {sp.latex(sp.factor(g))}
        rf" = {sp.latex(F)} + C"
    ))
    return F

F = sin_cos_odd(2, 5)
check(F, sp.sin(x)**2 * sp.cos(x)**5);
F = sin_cos_odd(3, 3)
check(F, sp.sin(x)**3 * sp.cos(x)**3);
```

$$u = \sin(x): \quad \int \sin^2 x \cos^5 x \, dx = \int u^2 (u-1)^2 (u+1)^2 \, du = \frac{\sin^7(x)}{7} - \frac{2\sin^5(x)}{5} + \frac{\sin^3(x)}{3} + C$$

```
d/dx(answer) == integrand: True
```

$$u = \sin(x): \quad \int \sin^3 x \cos^3 x \, dx = \int -u^3 (u-1)(u+1) \, du = -\frac{\sin^6(x)}{6} + \frac{\sin^4(x)}{4} + C$$

```
d/dx(answer) == integrand: True
```

Example 3: $\displaystyle\int_0^{\pi} \sin^2 x \, dx$ **(both powers even)**

Saving one $\sin x$ leaves $\sin x$, which can't be written in terms of $\cos x$ without a square root. When **both powers are even**, use the half-angle identity instead:

$$\int_0^{\pi} \sin^2 x \, dx = \tfrac{1}{2} \int_0^{\pi} (1 - \cos 2x) \, dx = \tfrac{1}{2} \Big[x - \tfrac{1}{2} \sin 2x \Big]_0^{\pi} = \tfrac{1}{2} \Big(\pi - \tfrac{1}{2} \sin 2\pi \Big) - \tfrac{1}{2}(0 - 0) = \frac{\pi}{2}.$$

Symbolic and numerical check:

```
exact = sp.integrate(sp.sin(x)**2, (x, 0, sp.pi))
display(Math(r"\int_0^\pi\sin^2x\,dx = " + sp.latex(exact) + rf" \approx {float(exact):.10f}"))

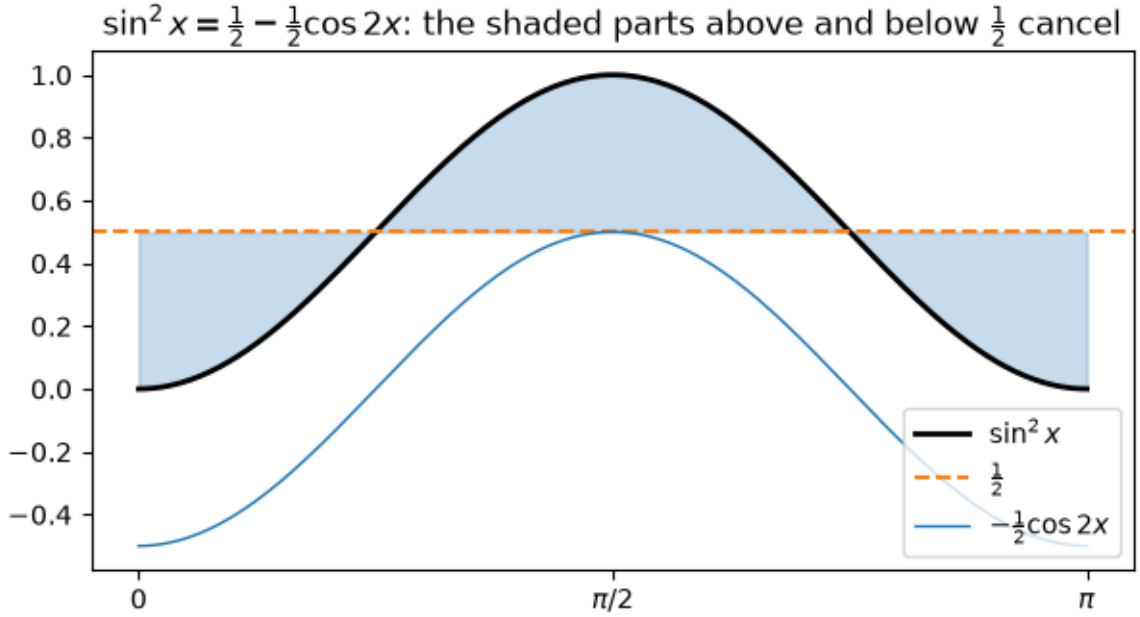
xs = np.linspace(0, np.pi, 100_001)
print(f"NumPy trapezoid: {np.trapezoid(np.sin(xs)**2, xs):.10f}")
```

$$\int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2} \approx 1.5707963268$$

```
NumPy trapezoid: 1.5707963268
```

Why $\pi/2$? The half-angle identity says $\sin^2 x$ is the constant $\frac{1}{2}$ plus a cosine wave $-\frac{1}{2}\cos 2x$. Over a whole period the wave has zero net area, so only the $\frac{1}{2}$ is left: $\frac{1}{2} \cdot \pi$.

```
t = np.linspace(0, np.pi, 400)
fig, ax = plt.subplots(figsize=(7, 3.5))
ax.plot(t, np.sin(t)**2, "k", lw=2, label=r"$\sin^2 x$")
ax.axhline(0.5, color="C1", ls="--", label=r"$\frac{1}{2}$")
ax.plot(t, -0.5 * np.cos(2 * t), color="C0", lw=1, label=r"$-\frac{1}{2}\cos 2x$")
ax.fill_between(t, 0.5, np.sin(t)**2, alpha=0.25, color="C0")
ax.set_xticks([0, np.pi / 2, np.pi], ["0", r"$\pi/2$", r"$\pi$"])
ax.set_title(r"$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$: the shaded parts above and below $\frac{1]
ax.legend(loc="lower right")
plt.show()
```



Example 4: $\int \sin^4 x\,dx$

Use the half-angle identity, then use it **again** on $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$:

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \tfrac{1}{4}\int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.\end{aligned}$$

In §7.1 we got $\int \sin^4 x\,dx$ from the reduction formula. The answer looked different there, but the two antiderivatives have the same derivative:

```
F = sp.Rational(3, 8) * x - sp.sin(2 * x) / 4 + sp.sin(4 * x) / 32
display(Math(r"\int\sin^4x\,dx = " + sp.latex(F) + " + C"))
```

```
check(F, sp.sin(x)**4)

# §7.1 reduction formula result: -1/4 cos x sin^3 x - 3/8 cos x sin x + 3/8 x
F_reduction = -sp.cos(x) * sp.sin(x)**3 / 4 - sp.Rational(3, 8) * sp.cos(x) * sp.sin(x) + sp.Rational(3,
print("Half-angle answer - reduction answer =", sp.simplify(F - F_reduction))
```

$$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$$

```
d/dx(answer) == integrand: True
Half-angle answer - reduction answer = 0
```

The strategy when both powers are even

Use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x),$$

and sometimes $\sin x\cos x = \tfrac{1}{2}\sin 2x$. Repeat as needed: each step halves the powers and doubles the angle.

For example, $\int \sin^2 x\cos^2 x\,dx = \int (\tfrac{1}{2}\sin 2x)^2\,dx = \tfrac{1}{8}\int(1-\cos 4x)\,dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$:

```
F = x / 8 - sp.sin(4 * x) / 32
display(Math(r"\int\sin^2x\cos^2x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.sin(x)**2 * sp.cos(x)**2);
```

$$\int \sin^2 x\cos^2 x\,dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

```
d/dx(answer) == integrand: True
```

3. Powers of tangent and secant: $\int \tan^m x \sec^n x\,dx$

The same idea works here, using the two derivatives

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x\tan x.$$

Example 5: $\int \tan^6 x\sec^4 x\,dx$

Save $\sec^2 x$ for du (with $u = \tan x$) and convert the remaining $\sec^2 x$ with $\sec^2 x = 1+\tan^2 x$:

$$\int \tan^6 x\sec^4 x\,dx = \int \tan^6 x\,(1+\tan^2 x)\sec^2 x\,dx = \int u^6(1+u^2)\,du = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C.$$

```
G = sp.integrate(sp.expand(u**6 * (1 + u**2)), u)
F = G.subs(u, sp.tan(x))
display(Math(r"\int\tan^6x\sec^4x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.tan(x)**6 * sp.sec(x)**4);
```

$$\int \tan^6 x\sec^4 x\,dx = \frac{\tan^9(x)}{9} + \frac{\tan^7(x)}{7} + C$$

```
d/dx(answer) == integrand: True
```

Example 6: $\int \tan^5\theta\sec^7\theta\,d\theta$

Now the power of secant is odd, so we can't save $\sec^2\theta$ and still convert the rest. The power of **tangent** is odd, so save $\sec\theta\tan\theta$ for du (with $u=\sec\theta$) and convert the remaining $\tan^4\theta$ with $\tan^2\theta=\sec^2\theta-1$:

$$\int \tan^5\theta\sec^7\theta\,d\theta = \int (\sec^2\theta-1)^2\sec^6\theta\cdot\sec\theta\tan\theta\,d\theta = \int (u^2-1)^2u^6\,du$$
$$= \frac{\sec^{11}\theta}{11} - \frac{2\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C.$$

```
G = sp.integrate(sp.expand((u**2 - 1)**2 * u**6), u)
F = G.subs(u, sp.sec(x))
display(Math(r"\int\tan^5\theta\sec^7\theta\,d\theta = " + sp.latex(F.subs(x, sp.Symbol("theta"))) + " +
check(F, sp.tan(x)**5 * sp.sec(x)**7);
```

$$\int \tan^5\theta\sec^7\theta\,d\theta = \frac{\sec^{11}(\theta)}{11} - \frac{2\sec^9(\theta)}{9} + \frac{\sec^7(\theta)}{7} + C$$

```
d/dx(answer) == integrand: True
```

The strategy for $\tan^m x\sec^n x$

Case	Save	Convert the rest with	Substitute
power of sec is **even** ($n\geq 2$)	$\sec^2 x$	$\sec^2 x = 1+\tan^2 x$	$u=\tan x$
power of tan is **odd**	$\sec x\tan x$	$\tan^2 x = \sec^2 x - 1$	$u=\sec x$

The same rules work for $\cot^m x\csc^n x$, using $\frac{d}{dx}\cot x = -\csc^2 x$, $\frac{d}{dx}\csc x = -\csc x\cot x$ and $1+\cot^2 x = \csc^2 x$.

```
def tan_sec(m, n):
    #  tan^m x sec^n x dx when sec power is even (n >= 2) or tan power is odd
    if n % 2 == 0 and n >= 2:            # save sec^2 x, u = tan x
        sub, g = sp.tan(x), u**m * (1 + u**2)**((n - 2) // 2)
    elif m % 2 == 1 and n >= 1:          # save sec x tan x, u = sec x
        sub, g = sp.sec(x), (u**2 - 1)**((m - 1) // 2) * u**(n - 1)
    else:
        raise ValueError("no direct substitution: use identities and/or integration by parts")
    G = sp.integrate(sp.expand(g), u)
    F = G.subs(u, sub)
    display(Math(
        rf"u = {sp.latex(sub)}:\quad \int \tan^{{{m}}}x\sec^{{{n}}}x\,dx = \int {sp.latex(sp.factor(g))}
        rf" = {sp.latex(F)} + C"
    ))
    return F

F = tan_sec(2, 4)
check(F, sp.tan(x)**2 * sp.sec(x)**4);
F = tan_sec(3, 3)
check(F, sp.tan(x)**3 * sp.sec(x)**3);
```

$u=\tan(x):\quad \int \tan^2 x\sec^4 x\,dx = \int u^2(u^2+1)\,du = \frac{\tan^5(x)}{5} + \frac{\tan^3(x)}{3} + C$

```
d/dx(answer) == integrand: True
```

$u = \sec(x)$: $\displaystyle\int \tan^3 x \sec^3 x\, dx = \int u^2 (u-1)(u+1)\, du = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

```
d/dx(answer) == integrand: True
```

The other cases: $\int \tan x\, dx$ **and** $\int \sec x\, dx$

If the power of tangent is even and the power of secant is odd, neither rule applies. We need two basic integrals.

Tangent. Write $\tan x = \dfrac{\sin x}{\cos x}$ and substitute $u = \cos x$:

$$\int \tan x\, dx = -\int \frac{du}{u} = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Secant. Multiply top and bottom by $\sec x + \tan x$. The new numerator is the derivative of the denominator:

$$\int \sec x\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx = \ln|\sec x + \tan x| + C.$$

(In the code we drop the absolute values. That doesn't change the derivative.)

```
F_tan = sp.log(sp.sec(x))
F_sec = sp.log(sp.sec(x) + sp.tan(x))
display(Math(r"\int\tan x\,dx = " + sp.latex(F_tan) + " + C"))
check(F_tan, sp.tan(x))
display(Math(r"\int\sec x\,dx = " + sp.latex(F_sec) + " + C"))
check(F_sec, sp.sec(x));
```

$\displaystyle\int \tan x\, dx = \log(\sec(x)) + C$

```
d/dx(answer) == integrand: True
```

$\displaystyle\int \sec x\, dx = \log(\tan(x) + \sec(x)) + C$

```
d/dx(answer) == integrand: True
```

Example 7: $\displaystyle\int \tan^3 x\, dx$

The power of tangent is odd, but there is no $\sec x$ to save. Instead, convert $\tan^2 x = \sec^2 x - 1$ and split:

$$\int \tan^3 x\, dx = \int \tan x (\sec^2 x - 1)\, dx = \int \tan x \sec^2 x\, dx - \int \tan x\, dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C.$$

The first integral uses $u = \tan x$.

```
F = sp.tan(x)**2 / 2 - sp.log(sp.sec(x))
display(Math(r"\int\tan^3x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.tan(x)**3);
```

$\displaystyle\int \tan^3 x\, dx = -\log(\sec(x)) + \frac{\tan^2(x)}{2} + C$

```
d/dx(answer) == integrand: True
```

Example 8: $\int \sec^3 x\,dx$ **(integration by parts)**

Odd powers of secant alone need integration by parts. This is the cyclic case from §7.1. Take

$$u = \sec x, \quad dv = \sec^2 x\,dx \quad \Longrightarrow \quad du = \sec x \tan x\,dx, \quad v = \tan x.$$

Then, with $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

$\int \sec^3 x\,dx$ appears on both sides. Solve for it:

$$\boxed{\int \sec^3 x\,dx = \tfrac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + C}$$

In SymPy, treat $I = \int \sec^3 x\,dx$ as an unknown, as in §7.1:

```
I = sp.Symbol("I")
equation = sp.Eq(I, sp.sec(x) * sp.tan(x) - I + F_sec)
display(equation)
F = sp.solve(equation, I)[0]
display(Math(r"\int\sec^3x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.sec(x)**3);
```

$I = -I + \log(\tan(x) + \sec(x)) + \tan(x)\sec(x)$

$$\int \sec^3 x\,dx = \frac{\log(\tan(x) + \sec(x))}{2} + \frac{\tan(x)\sec(x)}{2} + C$$

```
d/dx(answer) == integrand: True
```

4. Products with different angles

Integrals like $\int \sin 4x \cos 5x\,dx$ show up in Fourier series. Use the **product-to-sum** identities:

$$\begin{aligned}\sin A \cos B &= \tfrac{1}{2}\big[\sin(A-B) + \sin(A+B)\big] \\ \sin A \sin B &= \tfrac{1}{2}\big[\cos(A-B) - \cos(A+B)\big] \\ \cos A \cos B &= \tfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]\end{aligned}$$

Each one comes from adding or subtracting the addition formulas for $\sin(A \pm B)$ or $\cos(A \pm B)$.

```
A, B = sp.symbols("A B")
product_to_sum = [
    (sp.sin(A) * sp.cos(B), (sp.sin(A - B) + sp.sin(A + B)) / 2),
    (sp.sin(A) * sp.sin(B), (sp.cos(A - B) - sp.cos(A + B)) / 2),
    (sp.cos(A) * sp.cos(B), (sp.cos(A - B) + sp.cos(A + B)) / 2),
]
for lhs, rhs in product_to_sum:
    ok = sp.simplify(sp.expand_trig(lhs - rhs)) == 0
    display(Math(sp.latex(lhs) + " = " + sp.latex(rhs) + rf"\quad\text{{: {ok}}}"))
```

$$\sin(A)\cos(B) = \frac{\sin(A-B)}{2} + \frac{\sin(A+B)}{2} \quad : \text{True}$$

$$\sin(A)\sin(B) = \frac{\cos(A-B)}{2} - \frac{\cos(A+B)}{2} \quad : \text{True}$$

$$\cos(A)\cos(B) = \frac{\cos(A-B)}{2} + \frac{\cos(A+B)}{2} \quad : \text{True}$$

Example 9: $\int \sin 4x \cos 5x \, dx$

With $A = 4x$, $B = 5x$:

$$\sin 4x \cos 5x = \tfrac{1}{2}\big[\sin(-x) + \sin 9x\big] = \tfrac{1}{2}(-\sin x + \sin 9x),$$

so

$$\int \sin 4x \cos 5x \, dx = \tfrac{1}{2}\Big(\cos x - \tfrac{1}{9}\cos 9x\Big) + C.$$

```
F = (sp.cos(x) - sp.cos(9 * x) / 9) / 2
display(Math(r"\int\sin 4x\cos 5x\,dx = " + sp.latex(F) + " + C"))
check(F, sp.sin(4 * x) * sp.cos(5 * x));
```

$$\int \sin 4x \cos 5x \, dx = \frac{\cos(x)}{2} - \frac{\cos(9x)}{18} + C$$

```
d/dx(answer) == integrand: True
```

Application: orthogonality

For positive integers m and n, the product-to-sum identities give

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

When $m \neq n$, both $\cos(m-n)x$ and $\cos(m+n)x$ go through whole periods on $[-\pi, \pi]$, so their integrals are 0. When $m = n$, $\cos(m-n)x = 1$ contributes $\frac{1}{2} \cdot 2\pi = \pi$.

This fact is the basis of Fourier series. Here is the table of values for $m, n = 1, \ldots, 5$, computed both ways:

```
N = 5
exact = sp.Matrix(N, N, lambda i, j: sp.integrate(sp.sin((i + 1) * x) * sp.sin((j + 1) * x), (x, -sp.pi,
display(Math(r"\left[\int_{-\pi}^{\pi}\sin mx\sin nx\,dx\right]_{m,n=1}^{5} = " + sp.latex(exact)))

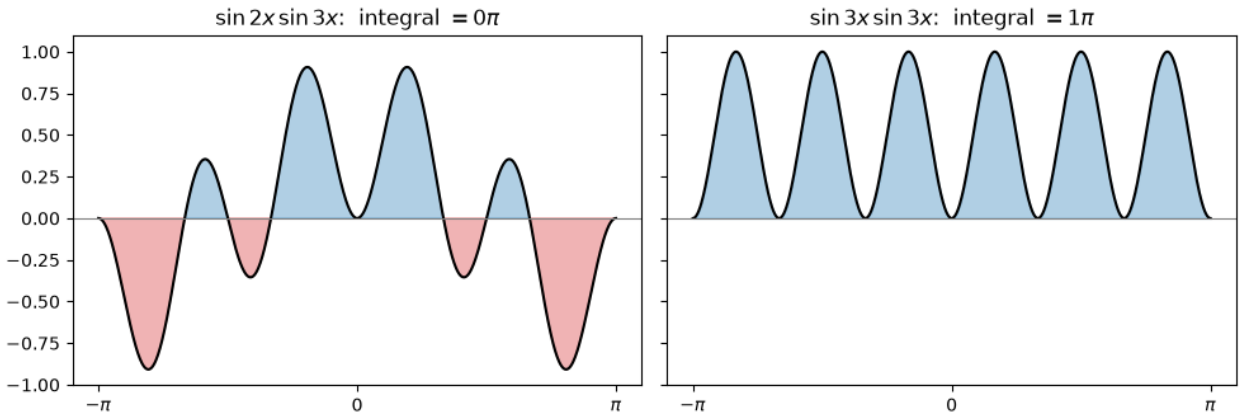
xs = np.linspace(-np.pi, np.pi, 20_001)
numeric = np.array([[np.trapezoid(np.sin(m * xs) * np.sin(k * xs), xs) for k in range(1, N + 1)] for m i
print("NumPy trapezoid (divided by ):")
print(np.round(numeric / np.pi, 8) + 0.0)  # + 0.0 turns -0. into 0.
```

$$\left[\int_{-\pi}^{\pi} \sin mx \sin nx \, dx\right]_{m,n=1}^{5} = \begin{bmatrix} \pi & 0 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 & 0 \\ 0 & 0 & \pi & 0 & 0 \\ 0 & 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & 0 & \pi \end{bmatrix}$$

```
NumPy trapezoid (divided by ):
[[1. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0.]
 [0. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 1.]]
```

```
t = np.linspace(-np.pi, np.pi, 800)
fig, axes = plt.subplots(1, 2, figsize=(10, 3.5), sharey=True)
for ax, (m, k) in zip(axes, [(2, 3), (3, 3)]):
```

```
    y = np.sin(m * t) * np.sin(k * t)
    ax.plot(t, y, "k", lw=1.5)
    ax.fill_between(t, 0, y, where=y > 0, alpha=0.35, color="C0")
    ax.fill_between(t, 0, y, where=y < 0, alpha=0.35, color="C3")
    ax.axhline(0, color="gray", lw=0.8)
    ax.set_xticks([-np.pi, 0, np.pi], [r"$-\pi$", "0", r"$\pi$"])
    ax.set_title(rf"$\sin {m}x\,\sin {k}x$:  integral $= {np.trapezoid(y, t) / np.pi:.0f}\pi$")
plt.tight_layout()
plt.show()
```



5. Common mistakes

1. **Saving the wrong factor.** To substitute $u = \sin x$ you must save a $\cos x$, and the other way around. With $u = \tan x$ save $\sec^2 x$; with $u = \sec x$ save $\sec x \tan x$.
2. **Using $u = \sin x$ when the power of $\cos x$ is even.** The leftover $\cos x$ becomes $\sqrt{1-u^2}$. Use the half-angle identities instead.
3. **Sign errors with $u = \cos x$:** $du = -\sin x\,dx$.
4. **Half-angle identities:** the angle doubles. $\int \cos^2 3x\,dx = \frac{1}{2}\int(1+\cos 6x)\,dx$, not $\cos 3x$ or $\cos 2x$.
5. **Forgetting the absolute values** in $\ln|\sec x|$ and $\ln|\sec x + \tan x|$.
6. **Different-looking answers.** Trig antiderivatives can differ by a constant, such as $\frac{1}{2}\sin^2 x$ and $-\frac{1}{2}\cos^2 x$ for $\int \sin x \cos x\,dx$. Differentiate to compare them.

```
F1, F2 = sp.sin(x)**2 / 2, -sp.cos(x)**2 / 2
check(F1, sp.sin(x) * sp.cos(x))
check(F2, sp.sin(x) * sp.cos(x))
print("F1 - F2 =", sp.simplify(F1 - F2))
```

```
d/dx(answer) == integrand: True
d/dx(answer) == integrand: True
F1 - F2 = 1/2
```

6. Practice problems

Try each by hand first, then run the cell below to check your answers. SymPy may give an answer that looks different from yours, so differentiate yours and compare.

1. $\displaystyle\int \sin^3 x \cos^2 x\,dx$
2. $\displaystyle\int \cos^4 x\,dx$

3. $\int \tan^2 x \sec^4 x \, dx$
4. $\int \tan^3 x \sec x \, dx$
5. $\int \cos 3x \cos 7x \, dx$
6. $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$
7. $\int_0^{\pi/4} \sec^4 x \, dx$

```
problems = [
    sp.sin(x)**3 * sp.cos(x)**2,
    sp.cos(x)**4,
    sp.tan(x)**2 * sp.sec(x)**4,
    sp.tan(x)**3 * sp.sec(x),
    sp.cos(3 * x) * sp.cos(7 * x),
]
for p in problems:
    F = sp.simplify(sp.integrate(p, x))
    display(Math(r"\int " + sp.latex(p) + r"\,dx = " + sp.latex(F) + " + C"))

definite = [
    (sp.sin(x)**2 * sp.cos(x)**2, 0, sp.pi / 2, lambda t: np.sin(t)**2 * np.cos(t)**2),
    (sp.sec(x)**4, 0, sp.pi / 4, lambda t: 1 / np.cos(t)**4),
]
for p, a, b, fn in definite:
    val = sp.integrate(p, (x, a, b))
    xs = np.linspace(float(a), float(b), 100_001)
    display(Math(rf"\int_{{{sp.latex(a)}}}^{{{sp.latex(b)}}} " + sp.latex(p) + r"\,dx = "
                 + sp.latex(sp.simplify(val)) + rf" \approx {float(val):.6f}"))
    print(f"NumPy check: {np.trapezoid(fn(xs), xs):.6f}")
```

$$\int \sin^3(x)\cos^2(x)\,dx = \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C$$

$$\int \cos^4(x)\,dx = \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$$

$$\int \tan^2(x)\sec^4(x)\,dx = \frac{\sin^3(x)}{3\cos^5(x)} - \frac{2\tan^5(x)}{15} + C$$

$$\int \tan^3(x)\sec(x)\,dx = \frac{-3 + \frac{1}{\cos^2(x)}}{3\cos(x)} + C$$

$$\int \cos(3x)\cos(7x)\,dx = \frac{\sin(4x)}{8} + \frac{\sin(10x)}{20} + C$$

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^2(x)\,dx = \frac{\pi}{16} \approx 0.196350$$

```
NumPy check: 0.196350
```

$$\int_0^{\frac{\pi}{4}} \sec^4(x)\,dx = \frac{4}{3} \approx 1.333333$$

```
NumPy check: 1.333333
```

Summary

Integrand	Strategy	Substitution
$\sin^m x \cos^n x$, n odd	Save $\cos x$, use $\cos^2 x = 1 - \sin^2 x$	$u = \sin x$
$\sin^m x \cos^n x$, m odd	Save $\sin x$, use $\sin^2 x = 1 - \cos^2 x$	$u = \cos x$
$\sin^m x \cos^n x$, both even	Half-angle identities, repeated	none
$\tan^m x \sec^n x$, n even	Save $\sec^2 x$, use $\sec^2 x = 1 + \tan^2 x$	$u = \tan x$
$\tan^m x \sec^n x$, m odd	Save $\sec x \tan x$, use $\tan^2 x = \sec^2 x - 1$	$u = \sec x$
$\tan^m x$ alone, odd $\sec^n x$	$\tan^2 x = \sec^2 x - 1$; parts for $\sec^3 x$	varies
$\sin mx \cos nx$, etc.	Product-to-sum identities	none

$$\int \tan x \, dx = \ln|\sec x| + C, \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$